*Journal of Global Optimization* **16:** 95–105, 2000.



# The Generic Stability and Existence of Essentially Connected Components of Solutions for Nonlinear Complementarity Problems

GEORGE ISAC[1] and GEORGE XIAN-ZHI YUAN[2,⋆]
[1]*Department of Mathematics and Computers, Royal Military College of Canada, Kingston, Ontario, Canada K7K 7B4;* [2]*Department of Mathematics, The University of Queensland, Brisbane, QLD, Australia 4072. E-mail: xzy@maths.uq.edu.au*

**Abstract.** The aim of this paper is to develop the general generic stability theory for nonlinear complementarity problems in the setting of infinite dimensional Banach spaces. We first show that each nonlinear complementarity problem can be approximated arbitrarily by a nonlinear complementarity problem which is stable in the sense that the small change of the objective function results in the small change of its solution set; and thus we say that almost all complementarity problems are stable from viewpoint of Baire category. Secondly, we show that each nonlinear complementarity problem has, at least, one connected component of its solutions which is stable, though in general its solution set may not have good behaviour (i.e., not stable). Our results show that if a complementarity problem has only one connected solution set, it is then always stable without the assumption that the functions are either Lipschitz or differentiable.

**Key words:** Baire category, Essential point, Essentially component, Generic stability, Nonlinear complementarity problem, Strong Karamardian's condition.

## 1. Introduction

The complementarity theory is dedicated to the study of complementarity problems - which is fundamental to the study of many optimization problems and the analysis and computation of equilibria in the physical and economic sense. It is well known that the complementarity theory has also many and remarkable applications in Engineering, Elasticity, Mechanics, Game Theory etc. The *solution set* of a complementarity problem can be *empty* or *non-empty*, *stable* or *unstable*. In this paper, our principle aim is to study the stability of solutions for nonlinear complementarity problems without the traditional assumptions such as *Lipschitz* or *differentiability* conditions on the functions by introducing a new concept called *essential solution* which reflects the stability of solutions for complementarity problems.

We recall that a general nonlinear complementarity problem in $\mathbb{R}^n$ is described as follows: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and $K \subset \mathbb{R}^n$ be an acute

⋆ Corresponding author.

convex closed cone with its vertex at the origin. Denoted by $K^*$ the dual cone for $K$, i.e., $K^* = \{y \in \mathbb{R}^n : \langle x, y\rangle \geqslant 0 \text{ for all } x \in K\}$. The general nonlinear complementarity problem (denoted by $GNCP(f, K)$) is to find a vector $x \in K$ such that

$$f(x) \in K^* \quad \text{and} \quad \langle x, f(x)\rangle = 0$$

where $\langle, \cdot, \rangle$ denotes the inner product of $\mathbb{R}^n$.

By introducing a notion called *exceptional family*, the characteristic for the existence of solutions for the generalized nonlinear problems has been established recently by Isac et al. [6, 7] (see also Bulavski et al. [1]). As applications, many general existence theorems of complementarity problems have been given which unify and improve corresponding existence theory of complementarity problems in the literature. For more details, see books of Hyers et al. [4] and Isac [5] and related references therein.

Throughout this paper, let $K$ be a non-trivial closed convex cone in a Banach space $E$ and $f : K \to E^*$ a continuous mapping unless specified. We denote by $SGNCP(f, K, D)$ the solution set which is contained in a non-empty subset $D$ of $K$ for the (nonlinear) complementarity problem $GNCP(f, K)$, i.e.,

$$SGNCP(f, K, D) := \{x \in D(\subset K) : f(x) \in K^* \text{ and } \langle x, f(x)\rangle = 0\}.$$

We first note that as a special case of Theorem 4.3.2 of Isac [5, p. 116], the following existence result for generalized nonlinear complementarity problem holds.

LEMMA 1.1 (Karamardian). *Let $K$ be a non-empty closed and convex cone of a Banach space $E$ and $D$ be a non-empty compact subset of $K$. Suppose $f : K \to E^*$ is a continuous mapping such that for each $x \in K \setminus D$, there exists $y \in D$ such that $\langle x - y, f(x)\rangle > 0$. Then all solutions of $GNCP(f, K)$ are contained in $D$, i.e., $SGNCP(f, K, D)$ is a non-empty and closed subset of $D$.*

## 2. The generic stability of nonlinear complementarity problems

The stability study of solutions of nonlinear complementarity problem is an important topic in complementarity theory. In this section, our aim is to develop the generic stability of solutions for generally nonlinear complementarity problem $GNCP(f, K)$ as introduced in Section 1 under the setting of infinite dimensional Banach spaces.

Let $K(E)$ be the space of all non-empty compact subsets of a metric space $(E, d)$ (e.g, the Banach space $(E, \|\cdot\|)$) equipped with the Hausdorff metric $h$ which is induced by the metric $d$ (resp., the norm $\|\cdot\|$). For any $\epsilon > 0$, $x_0 \in E$ and $A \in K(E)$, let $U(\epsilon, A) = \{x \in E : d(u, x) < \epsilon \text{ for some } u \in A\}$ and $O(x_0, \epsilon) = \{x \in E : d(x_0, y) < \epsilon\}$.

Let $Y$ be a topological space and we denote by $2^Y$ the family of all subsets of $Y$. We recall that a subset $Q \subset Y$ is called a *residual* set if it is a countable intersection of open dense subsets of $Y$. Let $X$ and $Y$ be two topological spaces and $F : X \to K(Y)$ be a set-valued mapping. Then $F$ is said to be usco if $F$ is upper semicontinous with non-empty and compact and convex values.

We now have the following result which is Theorem 2 of Fort [3]:

LEMMA 2.1. *Let X be a metric space, Y be a topological space and $F : Y \to K(X)$ an usco mapping. Then the set of points where $F$ is lower semicontinuous is a residual set in $Y$.*

LEMMA 2.2. *Let X be a metric space, Y be a complete metric space and $F : Y \to K(X)$ be an usco mapping. Then the set of points where $F$ is lower semicontinuous is a dense residual set in $Y$.*

*Proof.* Since $Y$ is complete, a residual set in $Y$ is dense; the result now follows from Lemma 2.1. □

Let $K$ be a non-empty closed and convex cone of a Banach space $(E, \|\cdot\|)$ and set $C := \{f : K \to E^* :$ and $f$ is continuous such that $\rho(f, f') := \sup_{x\in K} \|f(x) - f'(x)\|^*$ for each $f, f' \in C < \infty\}$, where $\|\cdot\|^*$ denotes the norm of the dual space $E^*$. Clearly, $\rho$ is a metric on $C$ and we have the following fact.

LEMMA 2.3. *The metric space $(C, \rho)$ is complete.*

Denoted by $K(K)$ the collection of all non-empty compact subsets of the cone $K$ in $(E, \|\cdot\|)$. Then we know that $K(K)$ is a complete metric space endowed with the Hausdorff metric $h$ (induced from the norm $\|\cdot\|$ of $E$).

Let $Y := C \times K(K)$ and we define a metric $d$ on $Y$ by $d(y, y') := \rho(f, f') + h(A, A')$ for each $y = (f, A)$ and $y' = (f', A') \in Y$. Then it is clear that $(Y, d)$ is also a complete metric space. Let $M := \{y = (f, A) \in Y$: such that $GNCP(f, K)$ has solutions in $A$, i.e., $SGNCP(f, K, A) \neq \emptyset\}$. Then we have the following result.

LEMMA 2.4. *The space $(M, d)$ is complete.*

*Proof.* Since $M \subset Y$ and $Y$ is complete, it is sufficient to prove that $M$ is closed in $Y$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $M$ and $y_n \to y \in Y$. Let $y_n := (f_n, A_n)$, $n = 1, 2, \ldots$ and $y = (f, A)$. Then $f_n \to f$ and $A_n \to A$. For each $n = 1, 2, \ldots$, since $y_n \in M$, there is $x_n \in A_n$ such that $f_n(x_n) \in K^*$ and $\langle x_n, f_n(x_n)\rangle = 0$. Since $A_n$ and $A$ are compact and $A_n \to A$, by A.5.1 (ii) of Mas-Colell [10, p.10], $\cup_{n=1}^{\infty} A_n \cup A$ is compact. Since $x_n \in A_n \subset \cup_{n=1}^{\infty} A_n \cup A$, without loss of generality we may assume that $x_n \to x \in \cup_{n=1}^{\infty} A_n \cup A$. If $x \notin A$, since $A$ is compact, there is $a > 0$ such that $U(a, A) \cap O(x, a) = \emptyset$. Since $A_n \to A$ and $x_n \to x$, there is $N_1$ such that $A_n \subset U(a, A)$ and $x_n \in O(x, a)$ for all $n \geqslant N_1$, which contradicts the assumption that $x_n \in A_n$. Hence we must have $x \in A$. By the fact that $K^*$ is closed, the continuity of $f$ and $\lim_{n\to\infty} f_n = f$, it is easy to see that $f(x) \in K^*$ and $\langle x, f(x)\rangle = 0$. Therefore $y = (f, A) \in M$ so that $M$ is closed in $Y$. □

For each $y = (f, A) \in M$, we denote by $S(y)$ the solution set of the nonlinear complementarity problem $GNCP(f, K)$ in $A$, i.e., $S(y) = SGNCP(f, K, A)$. Then we have that $S(y) \neq \emptyset$. In what follows, we shall also use $GNCP(f, K, A)$ to denote the generally nonlinear complementarity problem $GNCP(f, K)$ associated with solutions in a subset $A$ of the closed and convex cone $K$.

LEMMA 2.5. *The set $S(y) \in K(K)$ for each $y \in M$.*

*Proof.* The conclusion follows by the definition of $S(y)$, the continuity of $f$ and the fact that $(x_n, y_n) \to (x, y)$. Here we give its details as follows: Let $y = (f, A) \in M$ be given. Since that $x_n \to x \in \cup_{n=1}^{\infty} A_n \cup A$. If $x \notin A$, since $A$ is compact, there is $a > 0$ such that $U(a, A) \cap O(x, a) = \emptyset$. Since $A_n \to A$ and $x_n \to x$, there is $N_1$ such that $A_n \subset U(a, A)$ and $x_n \in O(x, a)$ for all $n \geqslant N_1$, which contradicts the assumption that $x_n \in A_n$. Hence we must have $x \in A$. Secondly, by the continuity of $f$, we must also have that $f(x) \in K^*$ and $\langle x, f(x)\rangle = 0$. Therefore $y = (f, A) \in M$ so that $M$ is closed in $Y$. This completes the proof. □

By Lemma 2.5, the mapping $y \mapsto S(y)$ defines a solution mapping $S : M \to K(K)$ and indeed we have the following upper semicontinuity of the mapping $S$.

LEMMA 2.6. *The solution mapping $S : M \to K(K)$ is upper semicontinuous on $M$.*

*Proof.* Suppose $S$ is not upper semicontinuous at $y \in M$, then there exist $\epsilon_0 > 0$ and a sequence $\{y_n\}_{n=1}^{\infty}$ in $M$ with $y_n \to y$ such that for each $n = 1, 2, \ldots,$ there exists $x_n \in S(y_n)$ with $x_n \notin U(\epsilon_0, S(y))$. Let $y_n = (f_n, A_n)$ and $y = (f, A)$, then $f_n \to f$ and $A_n \to A$. Since $x_n \in A_n \subset \cup_{n=1}^{\infty} A_n \cup A$ and $\cup_{n=1}^{\infty} A_n \cup A$ is compact, without loss of generality, we may assume that $x_n \to x \in \cup_{n=1}^{\infty} A_n \cup A$. Note that we must have $x \notin U(\epsilon_0, S(y))$. Now the same argument as in the proof of Lemma 2.4 shows that $x \in A$, $f(x) \in K^*$ and $\langle x, f(x)\rangle = 0$, so that $x \in S(y)$. This contradicts that $x \notin U(\epsilon_0, S(y))$. Therefore $S$ must be upper semicontinuous. This completes the proof. □

In order to study the stability of solution set for nonlinear complementarity problems, we now introduce the following notions.

DEFINITION 2.7. Let $M_1$ be a non-empty closed subset of $M$ (then $M_1$ is also complete as so is the space $M$). If $y = (f, A) \in M_1$, then a solution point $x$ in $S(y)$ is said to be an *essential solution* of the nonlinear complementarity problem $GNCP(f, K, A)$ with respect to $M_1$ provided that for any $\epsilon > 0$, there is $\delta > 0$ such that for any $y' = GNCP(f', A') \in M_1$ with $d(y, y') = \rho(f, f') + h(A, A') < \delta$, there exists a solution $x' \in S(y')$ for the complementarity problem $GNCP(f', K, A')$ with $\|x - x'\| < \epsilon$. The complementarity problem $GNCP$

$(f, K, A)$ is said to be *essential* (with respect to $M_1$) if every $x \in S(y)$ is an essential solution of $y$ with respect to $M_1$.

REMARK 2.1. Definition 2.7 says that if the complementarity problem $GNCP(f, K, A)$ is essential, then for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any other complementarity problem $GNCP(f', K, A')$ with $\sup_{x\in X}\{d(f(x), f'(x)) : x \in X\} + h(A, A') < \delta$, having at least on solution $x'$ in the $\epsilon$-neighbourhood of the solution set of $GNCP(f, K, A)$; or equivalently to saying, the solution mapping $S$ is continuous at the complementarity problem $GNCP(f, K, A)$ when it is essential. Therefore the essential property of $GNCP(f, K, A)$ characterizes the continuous property of its solution set in $A$.

Now we have the following characteristic of stability for solution set of complementarity problems.

THEOREM 2.8. *The solution mapping $S$ is lower semicontinuous at $y \in M_1$ if and only if $y$ is essential with respect to $M_1$.*

*Proof.* Suppose $S$ is lower semicontinuous at $y \in M_1$. Then for any $\epsilon > 0$, there is $\delta > 0$ such that for any $y' \in M_1$ with $d(y, y') < \delta$, we have $S(y) \subset U(\epsilon, S(y'))$ so that for any $x \in S(y)$, there is $x' \in S(y')$ with $d(x, x') < \epsilon$. Thus every $x \in S(y)$ is an essential solution of $y = GNCP(f, K)$ with respect to $M_1$ and hence $y$ $(= GNCP(f, k))$ is essential with respect to $M_1$.

Conversely, suppose that $y$ is essential with respect to $M_1$. If $S$ were not lower semicontinuous at $y \in M_1$, then there exist $\epsilon_0 > 0$ and a sequence $\{y_n\}_{n=1}^{\infty}$ in $M$ with $y_n \to y$ such that for each $n = 1, 2, \ldots,$ there is $x_n \in S(y)$ with $x_n \notin U(\epsilon_0, S(y_n))$. Since $S(y)$ is compact, we may assume that $x_n \to x \in S(y)$. Since $x$ is an essential solution of the nonlinear complementarity problem $y = GNCP(f, K)$ with respect to $M_1$, $y_n \to y$ and $x_n \to x$, there is $N$ such that $d(x_n, x) < \epsilon_0/2$ and $x \in U(\epsilon_0/2, S(y_n))$ for all $n \geqslant N$. Hence $x_n \in O(x, \epsilon_0/2) \subset U(\epsilon_0, S(y_n))$ for all $n \geqslant N$ which contradicts the assumption that $x_n \notin U(\epsilon_0, S(y_n))$ for all $n = 1, 2, \ldots$. Hence $S$ must be lower semicontinuous at $y$. □

The following theorem says that each complementarity problem can be arbitrarily approximated by an essential complementarity problem.

THEOREM 2.9. *The set of essential points with respect to $M_1$ is a dense residual set in $M_1$. In particular, every point in $M_1$ can be arbitrarily approximated by an essential point in $M_1$.*

*Proof.* By Lemma 2.5 and Lemma 2.6, $S : M \to K(K)$ is an usco mapping. Since $M_1$ is complete, by Lemma 2.2, the set of points where $S$ is lower semicontinuous is a dense residual set in $M_1$. By Theorem 2.8, the set of essential points in $M_1$ is a dense residual set in $M_1$. □

By combining Lemma 2.6, Theorems 2.8 and 2.9, we have the following result.

THEOREM 2.10. *The solution mapping S is continuous at $y \in M_1$ if and only if $y$ is essential with respect to $M_1$. Moreover, the set of points at which S is continuous is a dense residual set in $M_1$.*

We remark that $S$ is continuous at $y \in M_1$, if and only if for each $\epsilon > 0$, there is $\delta > 0$ such that $h(S(y), S(y')) < \epsilon$ for each $y' \in M$ with $d(y, y') < \delta$. Theorem 2.8 implies that if $y = (f, g, A) \in M_1$, then $y$ is essential with respect to $M_1$ if and only if its set $S(y)$ of solution points is stable: $S(y')$ is close to $S(y)$ whenever $y'$ is close to $y$.

We now give a sufficient condition that $y \in M_1$ is essential with respect to $M_1$:

THEOREM 2.11. *If $y \in M_1$ is such that $S(y)$ is a singleton set, then $y$ is essential with respect to $M_1$.*

*Proof.* Suppose $S(y) = \{x\}$. By Lemma 2.6 $S$ is upper semicontinuous at $y$. Thus for any $\epsilon > 0$, there is $\delta > 0$ such that for each $y' \in M_1$, $d(y, y') < \delta$ implies $S(y') \subset U(\epsilon, S(y)) = O(x, \epsilon)$ so that $S(y) = \{x\} \subset U(\epsilon, S(y'))$. This shows that $S$ is also lower semicontinuous at $y$. By Theorem 2.8, $y$ is essential with respect to $M_1$.

In this section, the generic stability results, mainly Theorems 2.9 and 2.10 tell us that though not all solution sets of complementarity problems have good behaviour, however there always exists some complementarity problem with stable solutions to approximate arbitrarily each of them, this indicates that almost all complementarity problems are stable in the sense of Baire category theory.

## 3. The existence of essentially connected components of solution set for nonlinear complementarity problems

As we have seen in last section, in general not all solutions of complementarity problems are stable though there exist complementarity problems with essential solutions to approximate them arbitrarily. In this section, however we will show that there exists at least one connected component of solution set for each complementarity problem, which is stable by introducing the concept of essential components of solution set for a class of complementarity problems which satisfy so-called *strong Karamardian's condition* (whose definition will be given below).

Suppose $x$ is a solution of the complementarity problem $y = GNCP(f, K, A) \in M$, then the component of the solution $x \in SGNCP(f, K, A)$ is the union of all connected subsets of $S(y)$ which contain the point $x$. From Engelking [2], we know that components are connected closed subsets of $S(y)$ and thus they are also compact as $S(y)$ is compact. It is also easy to see that the components of two distinct points of $S(y)$ either coincide or are disjoint, so that all components constitute a decomposition of $S(y)$ into connected pairwise disjoint compact subsets, i.e.,

$$S(y) = \bigcup_{\alpha \in \Lambda} S_\alpha(y)$$

where $\Lambda$ is an index set, for any $\alpha \in \Lambda$, $S_\alpha(y)$ is a nonempty connected compact and for any $\alpha, \beta \in \Lambda(\alpha \neq \beta)$, $S_\alpha(y) \cap S_\beta(y) = \emptyset$.

In order to study the existence of essentially connected components of generalized complementarity problems, we first introduce the following definition.

DEFINITION 3.1. For each complementarity problem $y = (f, K, A) \in M$, suppose the set $S(y) = \bigcup_{\alpha\in\Lambda} S_\alpha(y)$. Then a component $S_\alpha(y)$ for some $\alpha \in \Lambda$, is said to be an *essential component* of $y$ if for each open set $O$ containing $S_\alpha(y)$, there exists $\delta > 0$ such that for any other complementarity problem $y' = GNCP(f', K, A') \in Y$ with $d(y, y') = \rho(f, f') + h(A, A') < \delta$, we have that $S(y') \cap O \neq \emptyset$.

REMARK 3.1. Definition 3.1 above means that even though we could not expect the continuity for all solutions set of a given complementarity problem $y = GNCP(f, K, A)$, however, there is a case that maybe some component of its solution set enjoins the continuous stability. In the rest part of this paper, we will show that the existence of such nice component for each generally complementarity problem $GNCP(f, K, A)$.

We recall that for given non-empty subsets $A$ and $B$ of a metric space $E$, the Hausdorff metric $h$ between $A$ and $B$ is defined by $h(A, B) := \inf\{\epsilon : A \subset O(B, \epsilon) \text{ and } B \subset O(A, \epsilon)\}$.

In order to establish our existence theorem of essential components for solution set of complementarity problems, we first need the following result (see also Lemma 3.1 of Yu and Luo [11]).

LEMMA 3.2. *Let $A$, $B$ and $C$ be non-empty convex and bounded subsets of a normed linear space $E$. Then $h(A, \lambda B + \mu C) \leqslant \lambda h(A, B) + \mu h(A, C)$ where $h$ is the Hausdorff metric defined on $E$, $\lambda \geqslant 0$ and $\mu \geqslant 0$ with $\lambda + \mu = 1$.*

*Proof.* By the definition of Hausdorff metric $h(A, B)$, it suffices to prove that for any given $\epsilon_1 > 0$ and $\epsilon_2 > 0$ with $B \subset O(A, \epsilon_1)$ and $A \subset O(B, \epsilon_1)$, and $C \subset O(A, \epsilon_2)$ and $A \subset O(C, \epsilon_2)$, we have that $A \subset O(\lambda B + \mu C, \lambda\epsilon_1 + \mu\epsilon_2)$ and $\lambda B + \mu C \subset O(A, \lambda\epsilon_1 + \mu\epsilon_2)$. For any $a \in A$, as $A \subset O(B, \epsilon_1)$ and $A \subset O(C, \epsilon_2)$, there exist $b \in B$ and $c \in C$ such that $d(a, b) < \epsilon_1$ and $d(a, c) < \epsilon_2$. Note that $\lambda + \mu = 1$, it follows that

$$d(a, \lambda b + \mu c) = \|a - \lambda b - \mu c\| \leqslant \lambda\|a - b\| + \mu\|a - c\| \leqslant \lambda\epsilon_1 + \mu\epsilon_2,$$

which implies that $A \subset O(\lambda B + \mu C, \lambda\epsilon_1 + \mu\epsilon_2)$. By the convexity of $B$ and $C$ and the similar argument used above, we can also verify that $\lambda B + \mu C \subset O(A, \lambda\epsilon_1 + \mu\epsilon_2)$ and thus the proof is completed. □

In order to establish the general existence of essentially connected components of solution set for complementarity problems, we introduce some kind of Karamardian's condition (see Karamardian [8] and also Isac [5, pp. 116–117] and related references), which is called the *Strong Karamardian Condition* as follows:

A continuous function $f : K \to E^*$ is said to satisfy *Strong Karamardian's condition* on $K$ if there exists a compact subset $D$ of $K$ such that for each $x \in K \setminus D$, $\langle x - y, f(x)\rangle > 0$ for all $y \in D$.

Let $Y_1$ be the collection of all complementarity problems satisfying the strong Karamardian's condition, i.e.,

$$Y_1 := C_1(K) \times \{A \in K(K) : A \supset D\},$$

where $C_1(K)$ is the collection of all continuous mappings from $K$ to $E^*$ which satisfy the strong Karamardian's condition above with respect to the set $D$. It is clear that $Y_1 \subset Y$ and the solution set $\emptyset \neq SGNCP(y) \subset D$ for each $y = GNCP(f, K, A) \in Y_1$ by Lemma 1.1, and thus $Y_1$ is also a subset of $M$.

Now we have the following general existence result of essentially connected components of solution set for any complementarity problem from the class $Y_1$.

THEOREM 3.3. *Let $y = GNCP(f, K, A)$ be a given complementarity problem in $Y_1$. Then there exists at least one essentially connected component of the solution set $S(y)$.*

*Proof.* For any given $y = GNCP(f, K, A) \in Y_1$, suppose that the solution set $S(y)$ of the complementarity problem $GNCP(f, K, A)$ is decomposed as follows:

$$S(y) = \bigcup_{\alpha \in \Lambda} S_\alpha(y)$$

where $\Lambda$ is an index set, for any $\alpha \in \Lambda$, $S_\alpha(y)$ is a connected compact and for any $\alpha, \beta \in \Lambda(\alpha \neq \beta)$, $S_\alpha(x) \cap S_\beta(x) = \emptyset$. We shall prove that there exists at least one essential component of $S(y)$. Let us suppose otherwise there is no any essential connected component. Then for any $\alpha \in \Lambda$, there exists an open set $O_\alpha \supset S_\alpha(y)$ such that for any $\varepsilon > 0$, there is $y_\alpha \in Y$ with $\rho(y, y_\alpha) < \varepsilon$ such that $S(y_\alpha) \cap O_\alpha = \emptyset$. As $S(y)$ is compact, there exist two open and finite coverings $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$ which satisfy the following conditions:

(1) $\overline{W_i} \subset V_i$;

(2) $V_i \cap V_j = \emptyset$ for each $i \neq j$; and

(3) $V_i$ contains at least one $S_{\alpha_i}(y)$ with $O_{\alpha_i} \supset V_i \supset S_{\alpha_i}(y)$.

Indeed, by following Kinoshita [9], for each $\alpha \in \Lambda$, note that $S_\alpha(y)$ is connected and compact (thus regular) and $S_\alpha(y) \subset O_\alpha$, then there exists non-empty open subsets $V_\alpha$ and $W_\alpha$ of $O_\alpha$ such that $S_\alpha(y) \subset V_\alpha \subset \overline{V_\alpha} \subset W_\alpha \subset O_\alpha$. Note that $S_\alpha(y) \cap S_\beta(y) = \emptyset$ for each $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. Without loss of generality, we may also assume that $W_\alpha \cap W_\beta = \emptyset$ for each $\alpha, \beta \in \Lambda$. Then $\{V_\alpha\}_{\alpha \in \Lambda}$ and $\{W_\alpha\}_{\alpha \in \Lambda}$ are open coverings of $S(y)$. By the compactness of $S(y)$, it follows that there exists $n \in \mathbb{N}$ such that $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$ are open coverings of $S(y)$ which satisfy above conditions (1)–(3).

Now by Lemma 2.6, the solution mapping $S$ is upper semicontinuous at $y$ and $\bigcup_{i=1}^n W_i \supset S(y)$ and $\bigcup_{i=1}^n W_i$ is open, then there exists a $\delta > 0$ such that

$\bigcup_{i=1}^{n} W_i \supset S(y')$ for any $y' \in Y$ with $\rho(y, y') < \delta$. Thus there exists $y_{\alpha_i} \in Y$ with $\rho(y, y_{\alpha_i}) < \delta$ such that $S(y_{\alpha_i}) \cap O_{\alpha_i} = \emptyset$.

Let $y = GNCP(f, K, A)$ and $y_{\alpha_i} = GNCP((f_{\alpha_i}, K, A_{\alpha_i})$, where $i = 1, 2, \dots, n$. We define a mapping $f^* : K \to E^*$ by

$$f^*(x) = \begin{cases} f(x), & \text{if } x \in K \setminus \bigcup_{i=1}^{n} V_i, \\ f_{\alpha_i}(x), & \text{if } x \in \overline{W}_i \\ \lambda_i(x) f(x) + \mu_i(x) f_{\alpha_i}(x), & \text{if } x \in V_i \setminus \overline{W}_i. \end{cases}$$

where

$$\lambda_i(x) = \frac{d(x, \overline{W}_i)}{d(x, \overline{W}_i) + d(x, K \setminus \bigcup_{i=1}^{n} V_i)}$$

and

$$\mu_i(x) = \frac{d(x, K \setminus \bigcup_{i=1}^{n} V_i)}{d(x, \overline{W}_i) + d(x, K \setminus \bigcup_{i=1}^{n} V_i)}.$$

By the definition of $f^*$, it is easy to verify that $f^*$ is continuous and $f^*$ also satisfies the strong Karamardian condition, thus $y^* = GNCP(f^*, K, A) \in Y_1$. Therefore, $S(y^*) \neq \emptyset$. Note that $\rho(y, y_{\alpha_i}) < \delta$ for $i = 1, 2, \dots, n$, it follows by Lemma 3.2 that

$$h(f(x), \lambda_i(x) f(x) + \mu_i(x) f_{\alpha_i}(x)) \leqslant h(f(x), f_{\alpha_i}(x)).$$

Therefore, $\rho(y, y^*) < \delta$ and $S(y^*) \subset \bigcup_{i=1}^{n} W_i$. Note that for any $x_0 \in S(y^*)$, there is an index $i_0$ such that $x_0 \in W_{i_0}$, and hence $x_0 \in W_{i_0} \subset \overline{W}_{i_0} \subset O_{\alpha_{i_0}}$. Therefore, $f^*(x_0) = f_{\alpha_{i_0}}(x_0)$ and $x_0 \in S(y_{\alpha_{i_0}})$. This contradicts our assumption that $S(y_{\alpha_{i_0}}) \cap O_{\alpha_{i_0}} = \emptyset$. Hence there exists at least one essentially connected component of $S(y)$. This completes the proof. □

THEOREM 3.4. *If the complementarity problem $y = GNCP(f, K, A) \in Y_1$ is such that the solution set $S(y)$ of complementarity problem $GNCP(f, K, A)$ is either totally disconnected set, then $GNCP(f, K, A)$ is weakly essential. In particular, if the solution set of $GNCP(f, K, A)$ is either a singleton, or it is connected, then the problem $GNCP(f, K, A)$ is essential.*

*Proof.* Since $S(y)$ is a totally disconnected set, then $S(y) = \cup_{\alpha \in \Lambda} S_\alpha(\phi)$, where $S_\alpha(\phi)$ is a singleton set for each $\alpha \in \Lambda$. By Theorem 3.3, there exists $S_{\alpha_0}(\phi) = \{x_0\}$, which is an essential component of $S(y)$. It is clear that $x_0$ is essential and thus it is weakly essential. In the case, the solution set $S(y)$ is a singleton set or it is connected, then $y$ is is essential by Theorem 3.3 and thus the complementarity problem $GNCP(f, K, A)$ is essential and the proof is complete. □

REMARK 3.2. Theorems 3.2 and 3.3 tell us that each nonlinear complementarity problem has, at least, one connected component of its solutions which is stable though in general its solution set may not have a good behaviour (i.e., not stable). Theorem 3.4 tells us that if a complementarity problem has only one connected solution set, it must be stable. Here we don't need to require the function $f$ to be either Lipschitz or differentiable.

Finally we also note that by using the same idea used in this paper for the class of complementarity problems satisfying the strong Karamardian condition, the general existence results of essentially connected components of solutions for complementarity problems which satisfy such as *coercive*, or *weakly coercive condition*, or some other kinds of coercive conditions (see the book of Hyers et al. [4, p. 63], or Zeidler [12, p. 472] for their definitions) can be also established; and thus we omit all of their details here. Secondly, we can also establish the existence theory of essentially connected components of solutions sets for a class of nonlinear complementarity problem $GNCP(f, K)$ in which the function $f$ is without an *exceptional family* (for its definition, see Bulavski et al. [1], Hyers et al. [4] or Isac et al. [6]) when its solutions set is contained in a compact sets.

## 4. Conclusion

In this paper, by introducing notion of essential components, we establish the general generic stability theory for nonlinear complementarity problems in the setting of infinite dimensional Banach spaces. Our first result (i.e., Theorem 2.10) shows that each nonlinear complementarity problem can be approximated arbitrarily by a nonlinear complementarity problem which is stable in the sense that the small change of the objective function results in the small change of its solution set. This means almost all complementarity problems are stable from viewpoint of Baire category. Then Theorem 3.3 shows that each nonlinear complementarity problem has, at least, one connected component of its solutions which is stable, though in general its solution set may not have good behaviour. This means if a complementarity problem has only one connected solution set, it is then always stable without the assumption that the functions are either Lipschitz or differentiable.

## Acknowledgement

Both authors express their thanks to anonymous referees' useful report which led to the present version of this paper. G.X.-Y. Yuan is supported by the Australian Research Council.

## References

1. Bulavski, A.V., Isac, G. and Kalashnikov, V. (1998), Application of topological degree theory to complementarity problems, in A. Migdalas et al., (eds.), *Multilevel Optimization: Algorithms and Applications*, Kluwer Academic Publishers, Dordrecht, pp. 333–358.

2. Engelking, R. (1989), *General Topology*, revised and completed edition, Heldermann Verlag, Berlin.
3. Fort, M.K. Jr. (1951), Points of continuity of semicontinuous functions, *Publ. Math. Debrecen* 2: 100–102.
4. Hyers, D., Isac, G. and Rassias, Th.M. (1997), *Topics in Nonlinear Analysis and Applications*, World Scientific, Singapore, New Jersey.
5. Isac, G. (1992), *Complementarity Problems*, Lecture Notes in Maths, Springer-Verlag, Berlin, Vol. 1528.
6. Isac, I., Bulavski, A.V. and Kalashnikov, V. (1997), Exceptional families, topological degree and complementarity problems, *J. Global Optim*. 10: 207–225.
7. Isac, G. and Obuchowska W.T. (1998), Functions without exceptional families of elements and complementarity problems, *J. Optim. Theory Appl*. 99: 147–163.
8. Karamardian, S. (1971), Generalized complementarity problems, *J. Optim. Theory Appl*. 8: 161–168.
9. Kinoshita, S. (1952), On essential components of the set of fixed points, *Osaka Math. J.* 4: 19–22.
10. Mas-Colell, A. (1985), *The Theory of General Economic Equilibrium*, Cambridge University Press, Cambridge.
11. Yu, J. and Luo, Q. (1999), On essential components of the solution set of generalized games, *J. Math. Anal. Appl*. 230: 303–310.
12. Zeidler, E. (1985), *Nonlinear Functional Analysis and Its Applications. Vol. II/B: Nonlinear Monotone Operators*, Springer Verlag, New York.